Low-Thrust Many-Revolution Trajectory Optimization via Differential Dynamic Programming and a Sundman Transformation

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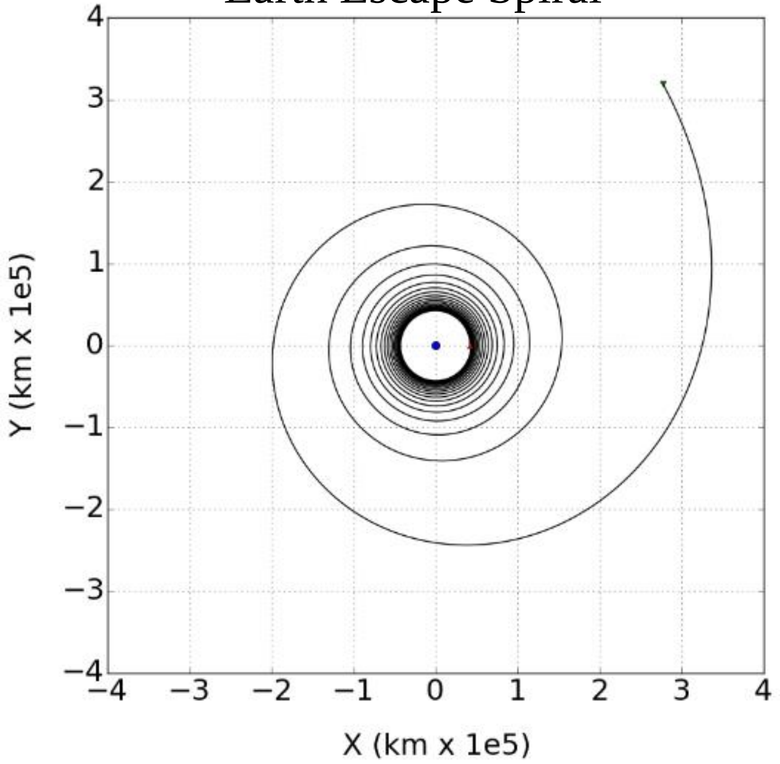
How many revolutions?

Planetary

- Long transfer durations with short orbital periods span many revolutions

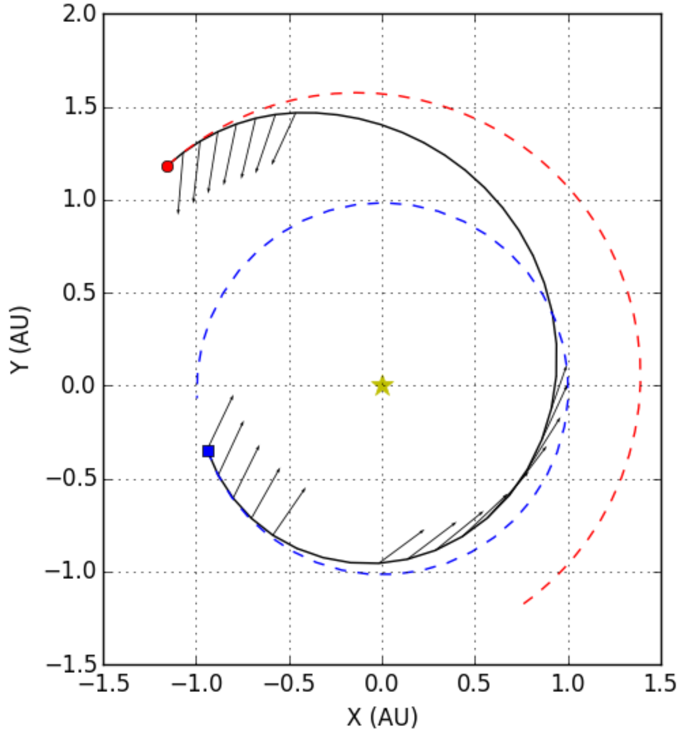
Earth Escape Spiral



Interplanetary

- Slow dynamics compared to control schedule

Earth-Mars Rendezvous



Number of ‘revs’: 10s, 100s, 1000s

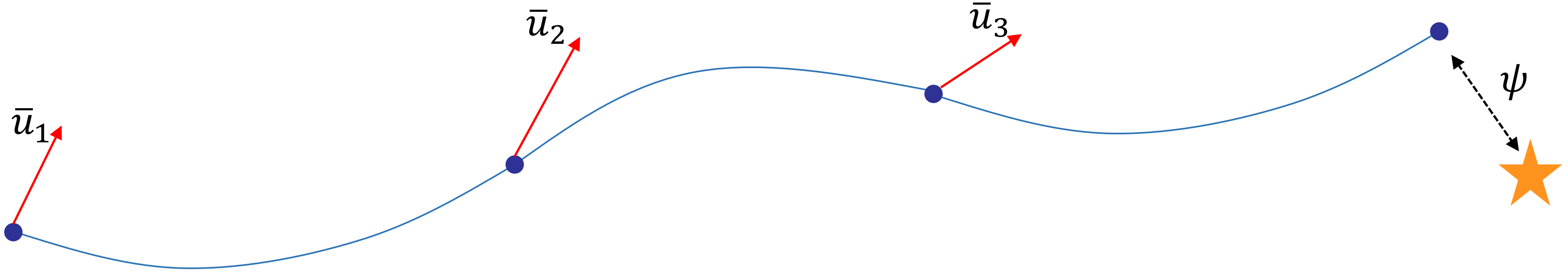
< 1, 1-10, 10s

Historical Approaches

Indirect	Control Laws	Direct
optimal control theory, apply Euler-Lagrange theorem and solve two point boundary value problem (TPBVP)	set a rule for spacecraft steering – a suboptimal policy that is acceptable by the mission designer	transcribe the trajectory optimization into a parameter optimization problem
Edelbaum Alfano Kéchichian	Kluever Chang Petropolous	Betts Whiffen Lantoine

HDDP

- Hybrid Differential Dynamic Programming
  - introduced by Lantoine and Russell
  - sequence of control updates that minimize quadratic model of *cost-to-go*
  - map derivatives along trajectory with state transition matrix and tensor



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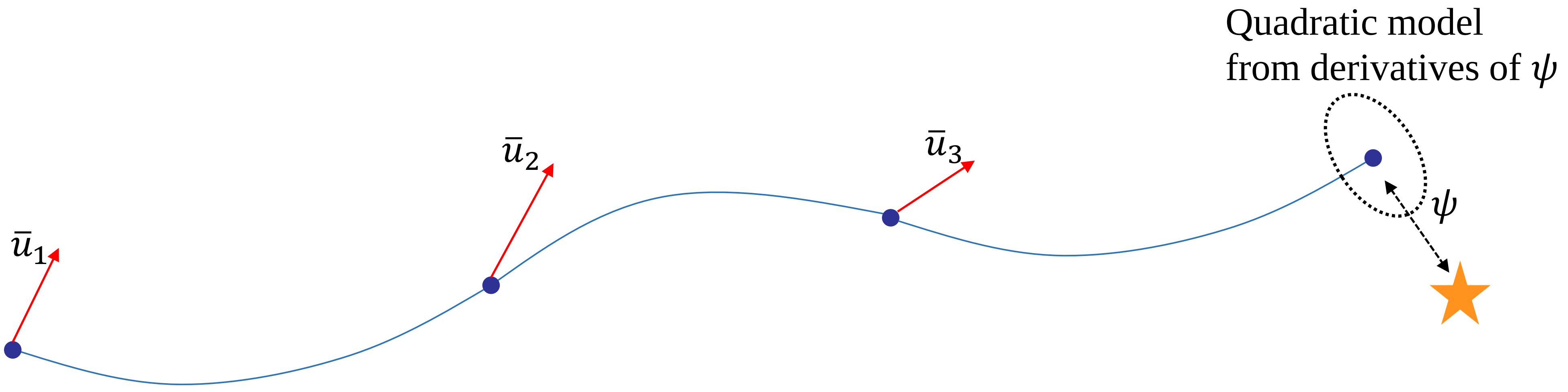

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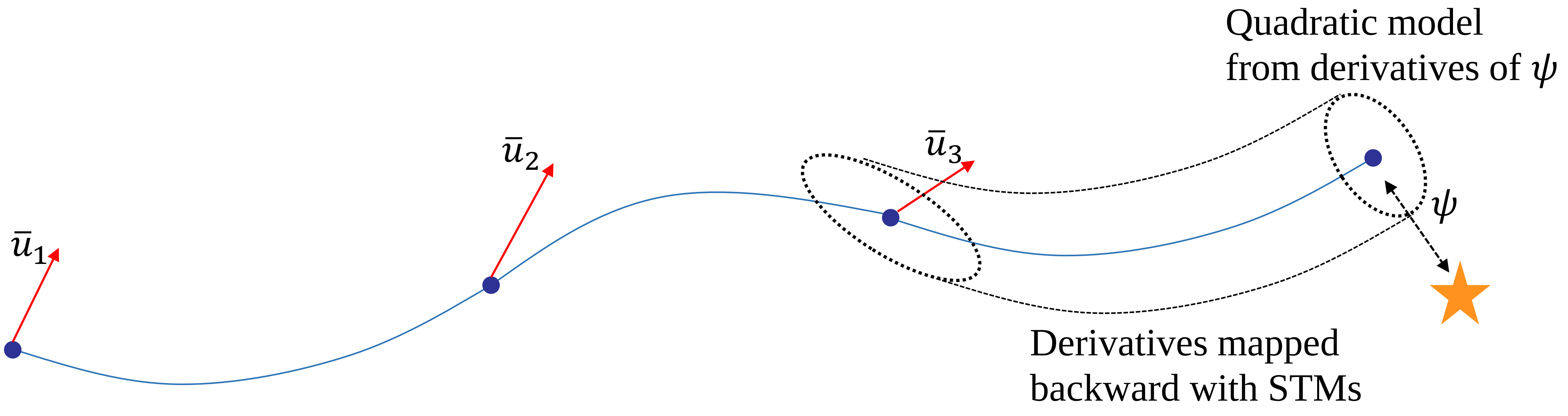

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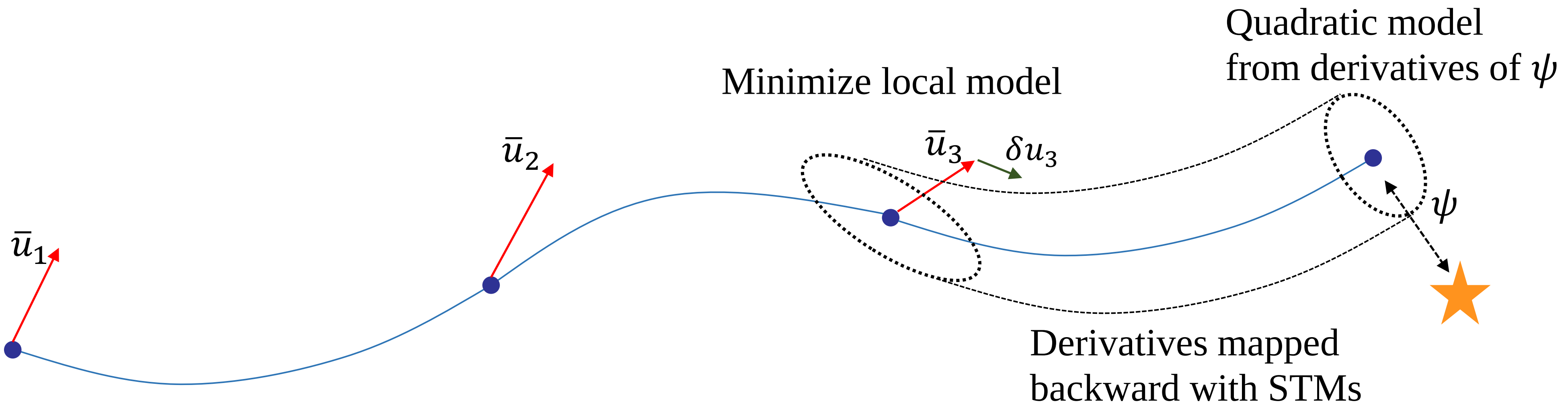

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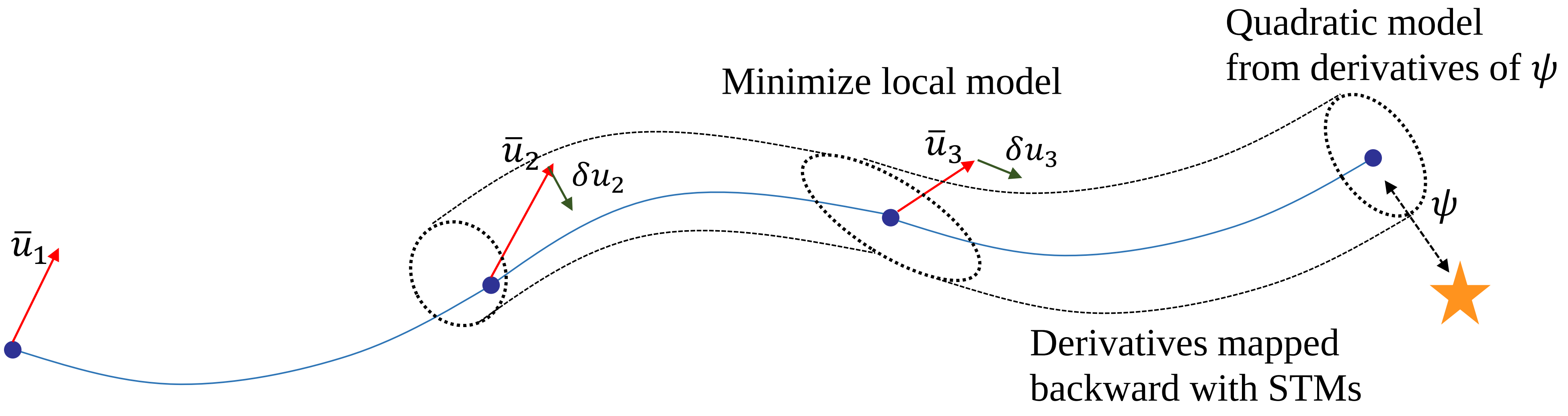

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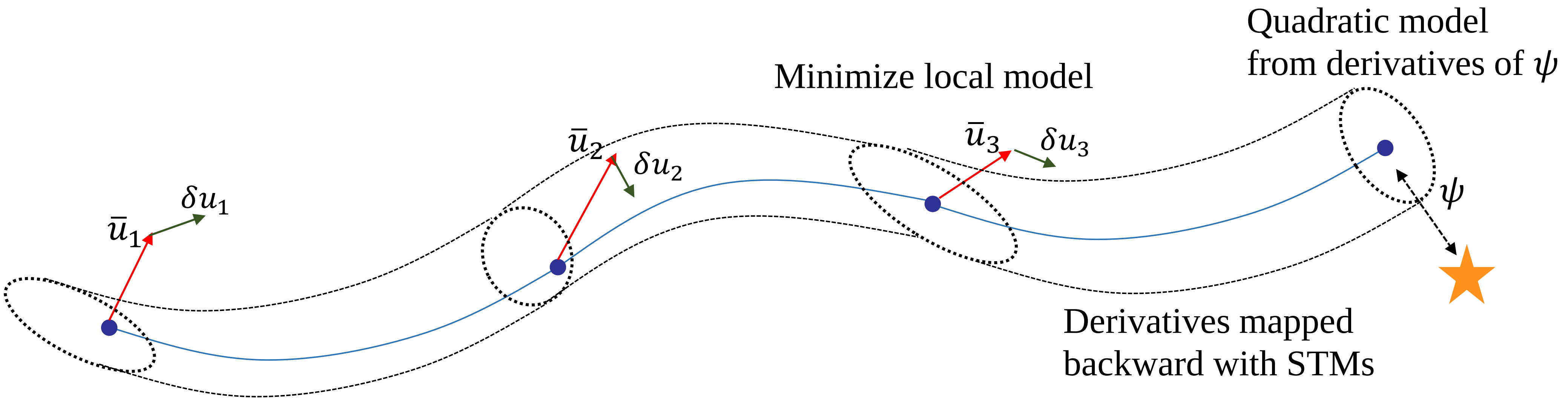

  - *Forward pass*: evaluate $\bar{u} + \delta u$ in equations of motion
  - *Backward sweep*: compute each δu_k

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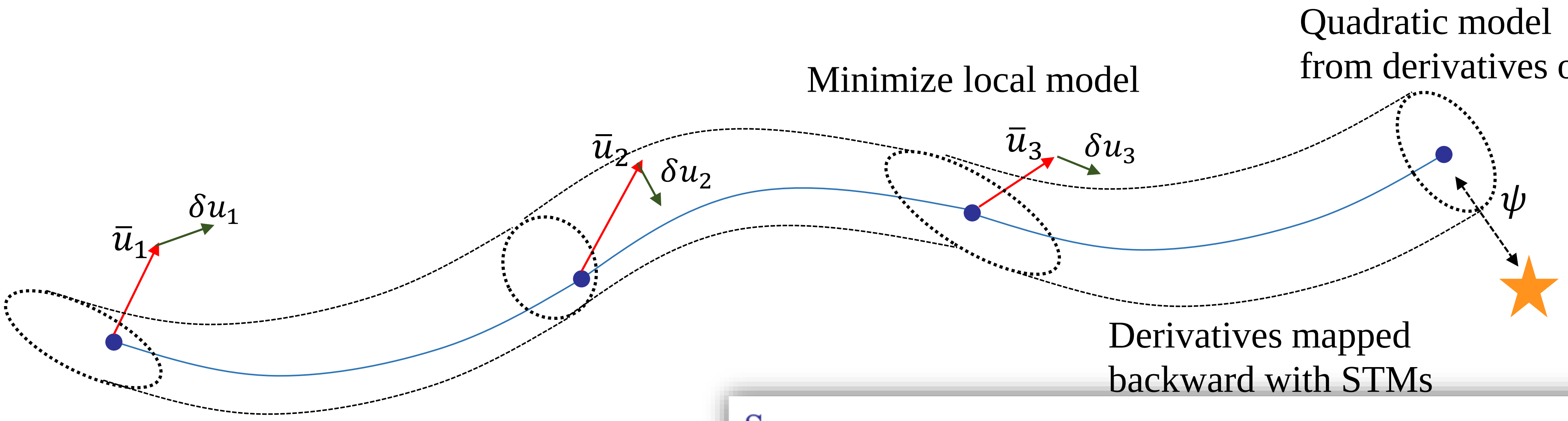


- *Forward pass*: evaluate $\bar{u} + \delta u$ i
- *Backward sweep*: compute each

See:
Gregory Lantoine and Ryan P. Russell. A hybrid differential dynamic programming algorithm for constrained optimal control problems. part 1: Theory. Journal of Optimization Theory and Applications, 154(2):382-417, 2012.

The Sundman Transformation

- Change independent variable from time to a function of orbital radius

$$dt = c_n r^n d\tau$$

- Can choose n, c_n, so that τ is an orbit angle

Eccentric Anomaly	Mean Anomaly	True Anomaly
$dt = \sqrt{\frac{a}{\mu}} r dE$	$dt = \sqrt{\frac{a^3}{\mu}} dM$	$dt = \frac{r^2}{h} d\nu$

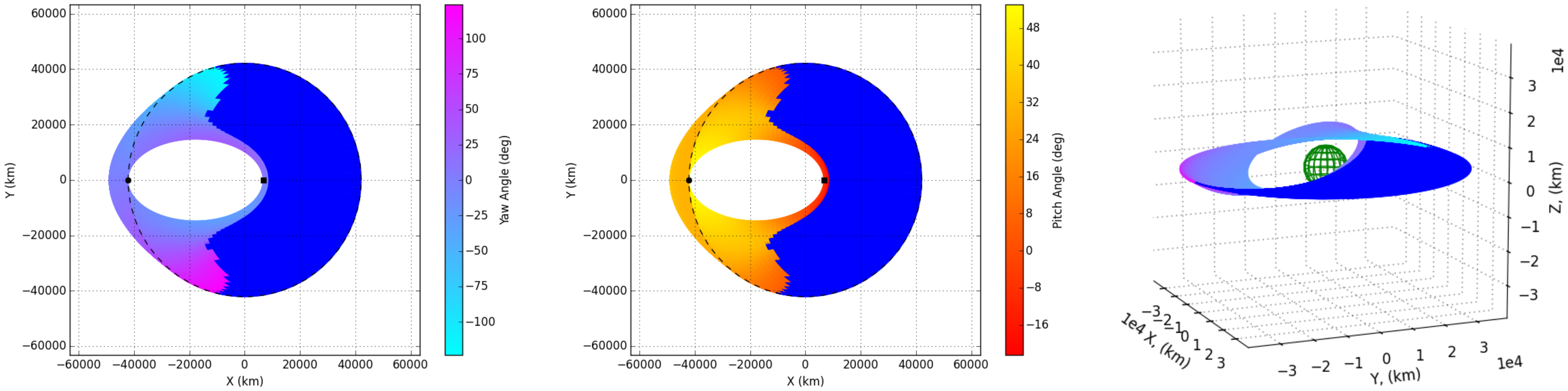
- Equations of motion become $x' = \dot{x} c_n r^n$
- Discretize in τ
- Specify τ_0, τ_f, rather than t_0, t_f
  - i.e. specify number of revolutions

- Track time in the state vector

$$x = \begin{bmatrix} t \\ x \\ y \\ z \\ \vdots \end{bmatrix}, \qquad x' = \begin{bmatrix} 1 \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \vdots \end{bmatrix} c_n r^n$$

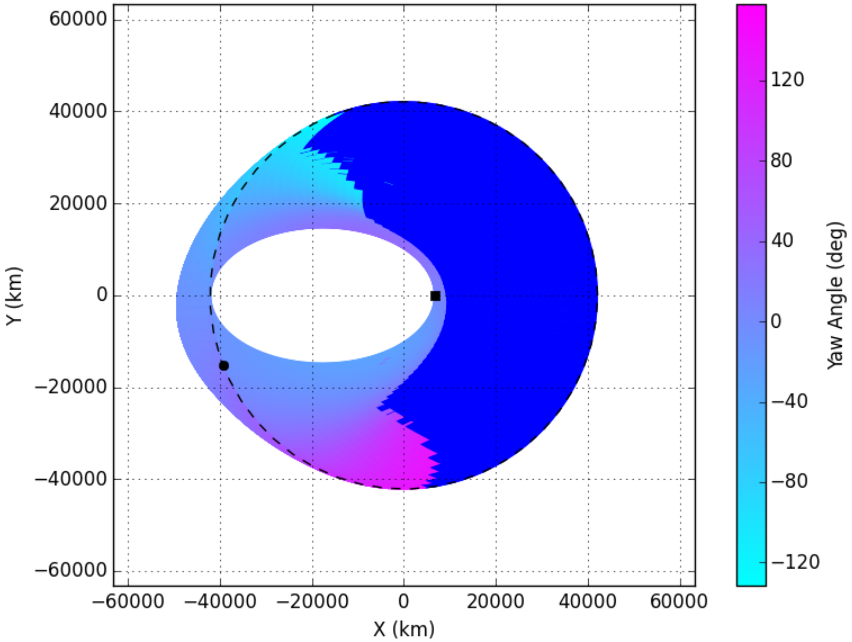
HDDP + Sundman

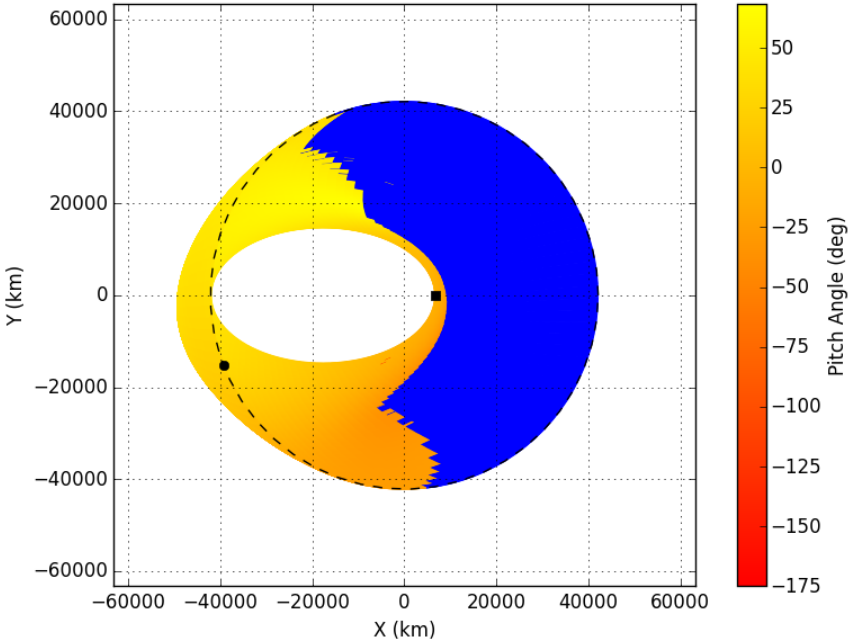
- Choose $\tau = E$, the eccentric anomaly
- Minimum fuel GTO to GEO in 450.5 revs
- 2-body dynamics
- 135,150 variables
- 54 minutes



HDDP + Sundman

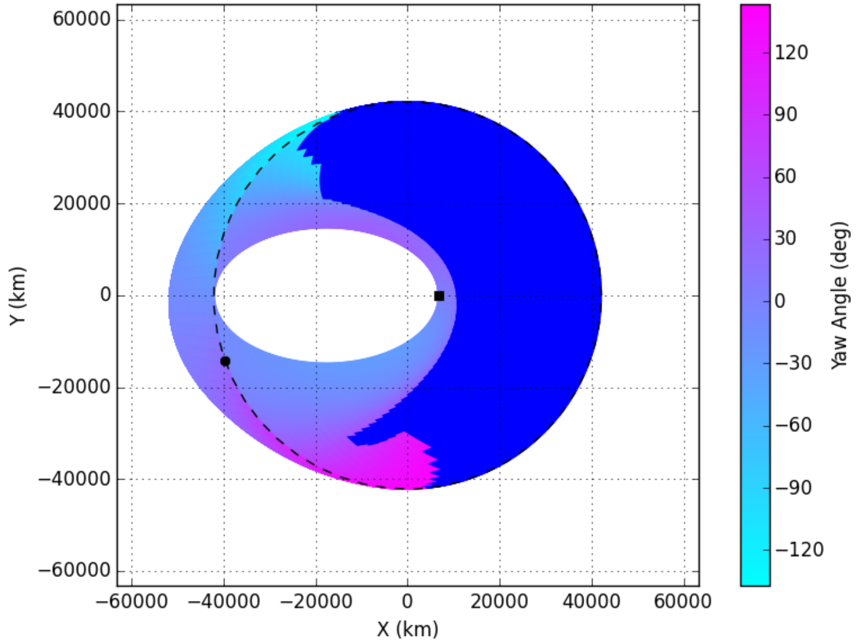
- Choose $\tau = E$, the eccentric anomaly
- Minimum fuel GTO to GEO in 450.5 revs
- 2-body dynamics + J_2
- 135,150 variables
- 61 minutes

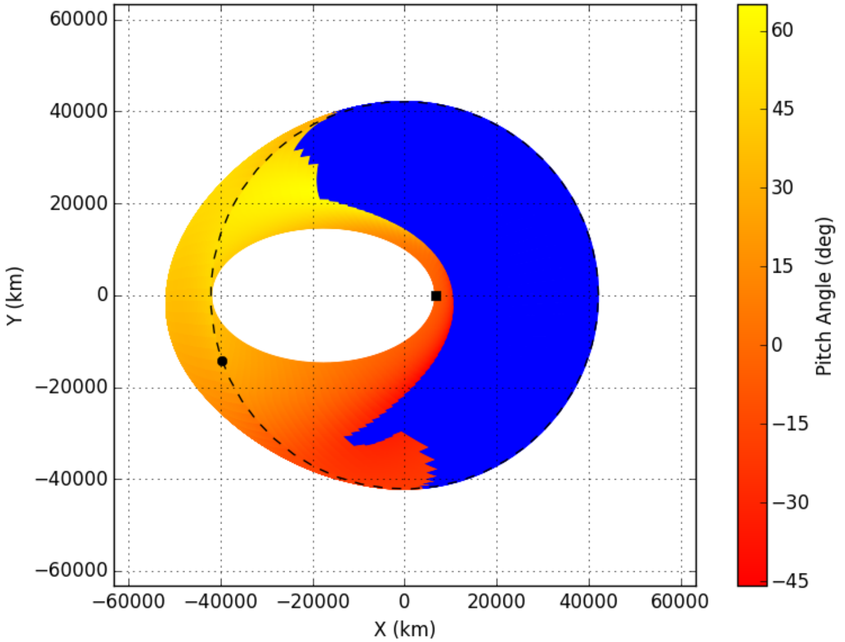




HDDP + Sundman

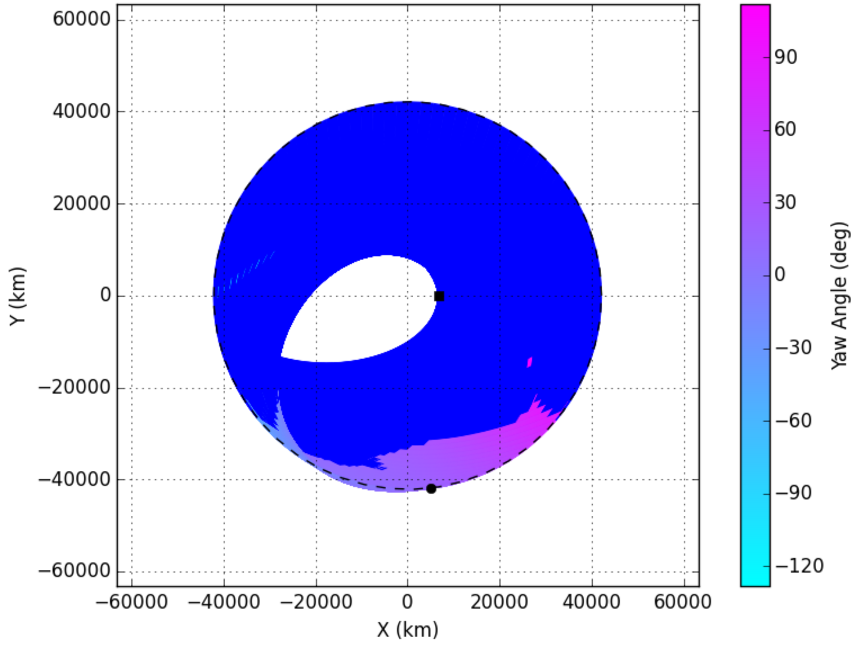
- Choose $\tau = E$, the eccentric anomaly
- Minimum fuel GTO to GEO in 450.5 revs
- 2-body dynamics + J_2+ lunar gravity
- 135,150 variables
- 107 minutes

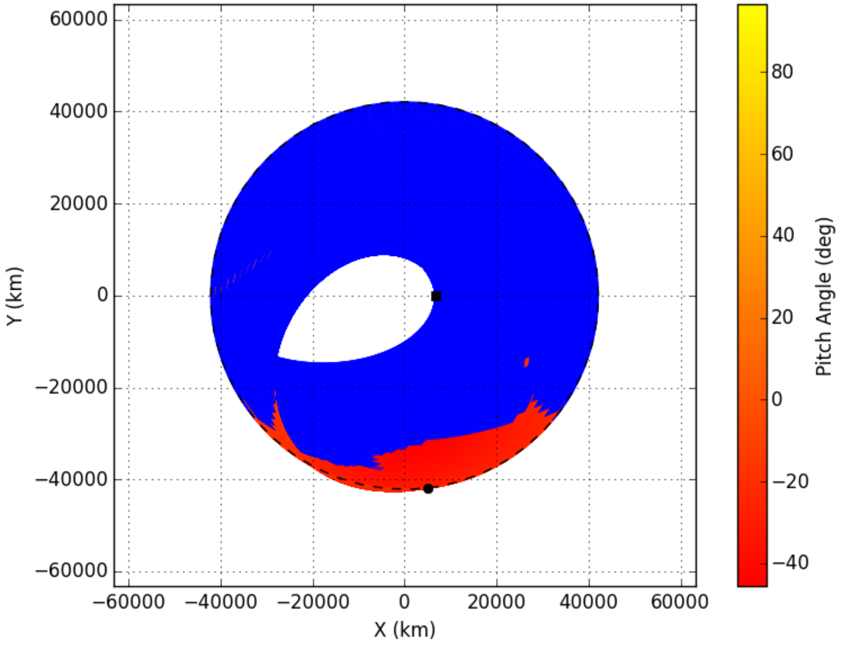




HDDP + Sundman

- Choose $\tau = E$, the eccentric anomaly
- Minimum fuel GTO to GEO in 1000.5 revs
- 2-body dynamics + J_2+ lunar gravity
- 300,150 variables
- 1359 minutes





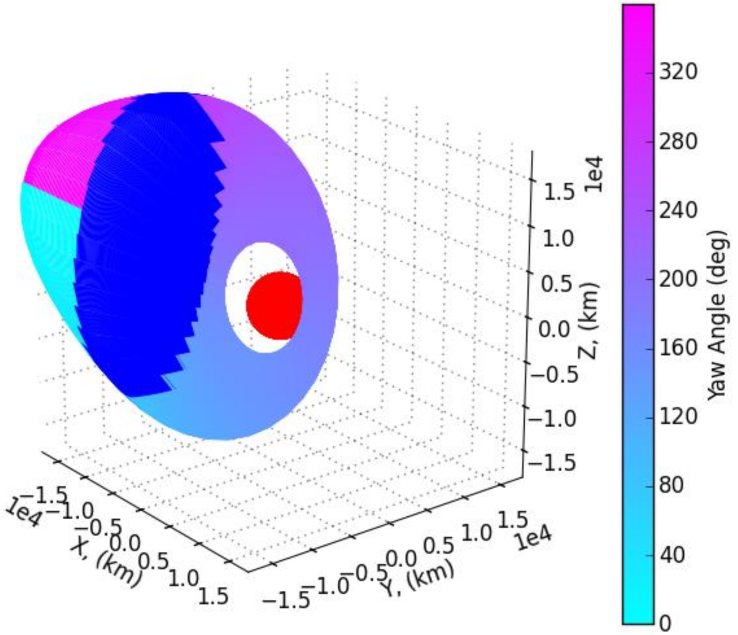
Backup Slides

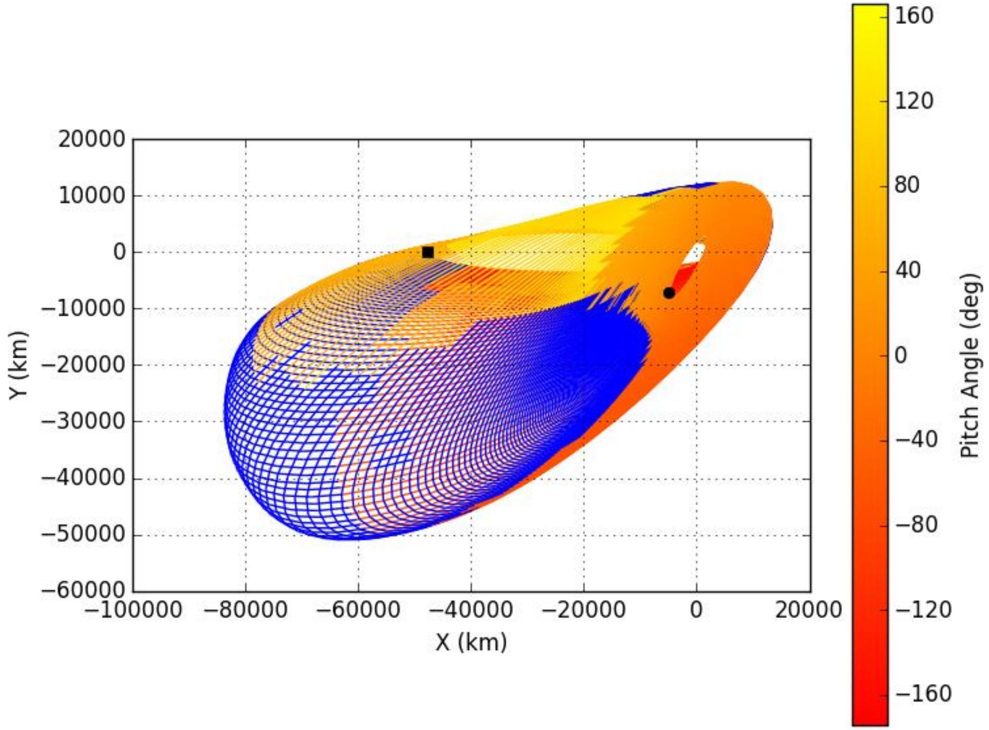
Table 2.1: Summary of GTO to GEO Results.

Perturbations	N_{rev}	Iterations	Runtime (minutes)	m_f (kg)	t_f (days)	θ (deg)
None	450.5	86	54	1759.1754	315.75	180.0
J_2	450.5	111	61	1737.1949	342.13	199.8728
J_2 and Lunar Gravity	450.5	136	107	1745.3012	322.63	201.0805
J_2 and Lunar Gravity	1000.5	913	1359	1784.3632	558.86	276.7209
None (Eclipse Model)	450.5	93	70	1751.4223	325.48	180.0
None (MEE)	450.5	59	15	1758.7230	318.96	180.0

HDDP + Sundman

500 rev orbit lowering at Mars





and with $\Delta\Omega = 60°$

Trust-Region Quadratic Subproblem

- Feedback control laws for δu, $\delta\lambda$, δw are unconstrained
- Likely to step beyond validity of quadratic expansion
- Require invertible, positive definite Hessians (negative definite for $J_{\lambda\lambda}$)
- Trust-region quadratic subproblem (TRQP):

$$\min_{\delta \boldsymbol{u}_k}[J_{u,k}^T \delta \boldsymbol{u}_k + \frac{1}{2}\delta \boldsymbol{u}_k^T J_{uu,k} \delta \boldsymbol{u}_k]$$

$$\text{s.t.} \quad \|D\delta \boldsymbol{u}_k\| \leqslant \Delta$$

- Acceptance of an iterate:

$$\rho = \frac{\delta J}{ER_{0,0}} \qquad \Delta_{p+1} = \begin{cases} \min((1+\kappa)\Delta_p, \Delta_{\max}), & \text{if } \rho \in [1-\epsilon_1, 1+\epsilon_1] \\ \max((1-\kappa)\Delta_p, \Delta_{\min}), & \text{otherwise} \end{cases}$$

$$\boldsymbol{X} = \begin{bmatrix} t & x & y & z & \dot{x} & \dot{y} & \dot{z} & m & T & \alpha & \beta \end{bmatrix}^T \tag{2}$$

$$\begin{bmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{s}} & \hat{\boldsymbol{w}} \end{bmatrix} = \begin{bmatrix} \dfrac{\boldsymbol{r}}{r} & \dfrac{(\boldsymbol{r} \times \boldsymbol{v}) \times \boldsymbol{r}}{\|(\boldsymbol{r} \times \boldsymbol{v}) \times \boldsymbol{r}\|} & \dfrac{\boldsymbol{r} \times \boldsymbol{v}}{\|\boldsymbol{r} \times \boldsymbol{v}\|} \end{bmatrix}. \tag{3}$$

$$\begin{bmatrix} T_r \\ T_s \\ T_w \end{bmatrix} = \begin{bmatrix} T \sin\alpha \cos\beta \\ T \cos\alpha \cos\beta \\ T \sin\beta \end{bmatrix}, \quad \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{s}} & \hat{\boldsymbol{w}} \end{bmatrix} \begin{bmatrix} T_r \\ T_s \\ T_w \end{bmatrix}, \tag{4}$$

$$\dot{\boldsymbol{X}} = \dot{\boldsymbol{X}}_{\oplus} + \dot{\boldsymbol{X}}_{T} + \dot{\boldsymbol{X}}_{J_2} + \dot{\boldsymbol{X}}_{☾}, \tag{5}$$

$$\dot{\boldsymbol{X}}_{\oplus} = \begin{bmatrix} 1 & \dot{x} & \dot{y} & \dot{z} & -\dfrac{\mu_{\oplus}}{r^3}x & -\dfrac{\mu_{\oplus}}{r^3}y & -\dfrac{\mu_{\oplus}}{r^3}z & 0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{6a}$$

$$\dot{\boldsymbol{X}}_{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dfrac{T_x}{m} & \dfrac{T_y}{m} & \dfrac{T_z}{m} & -\dfrac{T}{I_{sp} g_0} & 0 & 0 & 0 \end{bmatrix}^T, \tag{6b}$$

$$\dot{\boldsymbol{X}}_{J_2} = -\frac{3 J_2 \mu_{\oplus} R_{\oplus}{}^2}{2 r^5} \begin{bmatrix} 0 & 0 & 0 & 0 & x(1 - 5\dfrac{z^2}{r^2}) & y(1 - 5\dfrac{z^2}{r^2}) & z(3 - 5\dfrac{z^2}{r^2}) & 0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{6c}$$

$$\dot{\boldsymbol{X}}_{☾} = -\mu_{☾} \begin{bmatrix} 0 & 0 & 0 & 0 & \dfrac{x - x_{☾}}{\|\boldsymbol{r} - \boldsymbol{r}_{☾}\|^3} + \dfrac{x_{☾}}{r_{☾}{}^3} & \dfrac{y - y_{☾}}{\|\boldsymbol{r} - \boldsymbol{r}_{☾}\|^3} + \dfrac{y_{☾}}{r_{☾}{}^3} & \dfrac{z - z_{☾}}{\|\boldsymbol{r} - \boldsymbol{r}_{☾}\|^3} + \dfrac{z_{☾}}{r_{☾}{}^3} & 0 & 0 & 0 & 0 \end{bmatrix}^T. \tag{6d}$$

$$\mathring{\boldsymbol{X}} = (\dot{\boldsymbol{X}}_{\oplus} + \dot{\boldsymbol{X}}_{\tau} + \dot{\boldsymbol{X}}_{J_2} + \dot{\boldsymbol{X}}_{\leftmoon})\sqrt{a/\mu_{\oplus}}\, r \tag{15}$$

$$A^{i,j} = \frac{\partial \dot{\boldsymbol{X}}^i}{\partial \boldsymbol{X}^j}\,, \tag{16a}$$

$$A^{i,jk} = \frac{\partial^2 \dot{\boldsymbol{X}}^i}{\partial \boldsymbol{X}^j \partial \boldsymbol{X}^k}\,, \tag{16b}$$

$$\Lambda^{i,j} = \frac{\partial \mathring{\boldsymbol{X}}^i}{\partial \boldsymbol{X}^j} \tag{17a}$$

$$\Lambda^{i,jk} = \frac{\partial^2 \mathring{\boldsymbol{X}}^i}{\partial \boldsymbol{X}^j \partial \boldsymbol{X}^k} \tag{17b}$$

$$\eta = dt/d\tau = c_n r^n \tag{18}$$

$${\eta_X}^i = \frac{\partial \eta}{\partial \boldsymbol{X}^i} \tag{19}$$

$${\eta_{XX}}^{i,j} = \frac{\partial^2 \eta}{\partial \boldsymbol{X}^i \partial \boldsymbol{X}^j} \tag{20}$$

$$\Phi^{i,a}(t_k, t_{k+1}) = \frac{\partial \boldsymbol{X}^i_{k+1}}{\partial \boldsymbol{X}^a_k} = \boldsymbol{F}^{i,a}_{X,k}$$

$$\Phi^{i,ab}(t_k, t_{k+1}) = \frac{\partial^2 \boldsymbol{X}^i_{k+1}}{\partial \boldsymbol{X}^a_k \partial \boldsymbol{X}^b_k} = \boldsymbol{F}^{i,ab}_{XX,k}$$

$$\dot{\Phi}^{i,a} = A^{i,\gamma_1}\Phi^{\gamma_1,a}$$

$$\dot{\Phi}^{i,ab} = A^{i,\gamma_1}\Phi^{\gamma_1,ab} + A^{i,\gamma_1\gamma_2}\Phi^{\gamma_1,a}\Phi^{\gamma_2,b}$$

$$A^{i,a} = \frac{\partial \dot{\boldsymbol{X}}^i}{\partial \boldsymbol{X}^a}$$

$$A^{i,ab} = \frac{\partial^2 \dot{\boldsymbol{X}}^i}{\partial \boldsymbol{X}^a \partial \boldsymbol{X}^b}$$

$$\Lambda^{i,j} = A^{i,j}\eta + \dot{\boldsymbol{X}}^i {\eta_X}^j \tag{21a}$$

$$\Lambda^{i,jk} = A^{i,jk}\eta + A^{i,j}{\eta_X}^k + A^{i,k}{\eta_X}^j + \dot{\boldsymbol{X}}^i {\eta_{XX}}^{j,k} \tag{21b}$$

$$\mathring{\Phi}^{i,j} = \Lambda^{i,a}\Phi^{a,j} \tag{27a}$$

$$\mathring{\Phi}^{i,jk} = \Lambda^{i,a}\Phi^{a,jk} + \Lambda^{i,ab}\Phi^{a,j}\Phi^{b,k} \tag{27b}$$